Analytical prediction of Nusselt number in a simultaneously developing laminar flow between parallel plates

A. A. Rostami and S. S. Mortazavi
Department of Mechanical Engineering, Isfahan University of Technology, Isfahan, Iran

A simultaneously developing laminar flow between parallel plates was studied. The wall temperatures were uniform and equal. A linear profile was used for the axial component of the velocity and then the energy equation was solved by the method of similarity. The expression obtained for the Nusselt number in this manner contained terms involving wall shear stress. In order to eliminate these terms, the velocity profile was approximated by a second-degree polynomial, which finally led to a closed form expression for the Nusselt number in terms of x^* and Pr. The prediction was compared with more exact existing results, which indicated the accuracy of the present analysis.

Keywords: developing laminar flow; forced convection; parallel plates

Introduction

Laminar flow exists in the majority of compact heat exchangers because of their small hydraulic diameters. When the flow channel length is short, the effects of the entrance region, the heat transfer coefficient, and the friction factor cannot be neglected.

The Graetz solution, which assumes a hydrodynamically developed flow, is not suitable for gases or even liquids with a Prandtl number of less than 5 unless an appropriate unheated starting length is present. Moreover, the series solution of Graetz converges very slowly as x^* approaches zero. For example the first 121 terms of the series are insufficient to accurately determine the Nusselt number for $x^* < 10^{-4}$.[1] Lévêque[2] overcame this difficulty by employing the flat plate solution as an asymptotic approximation near the point where the step change in the wall temperature occurs.

For gases and liquids with a low Pr (say, less than 5) entering a flow channel with a wall temperature other than the fluid inlet temperature, a simultaneously developing flow will prevail. To solve this problem the velocity components (u, v) have to be determined from continuity and momentum differential equations. Among the methods used to solve these equations is the integral method first used by Schiller[3] for the circular tube and parallel plate channel. He used a parabolic velocity distribution in the boundary layer and Bernoulli's equation in the inviscid core to determine the pressure distribution in the axial direction. This method yields a discontinuous solution for the gradients of velocity and pressure distribution. To alleviate this problem, Langhaar[4] proposed a method of linearization for a circular tube. He linearized the nonlinear inertia terms of the momentum equation as follows:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\beta^2(x)u \tag{1}$$

Langhaar's approach was used by Han[5] for parallel plates.

The combined entry length problem for the circular tube was first investigated numerically by Kays.[6] He used the Langhaar velocity profile for axial velocity distribution and neglected the radial velocity component and axial heat conduction. Kays made calculations for Pr = 0.7. Later, Goldberg[7] extended Kays's solution to cover a range of Prandtl numbers from 0.5 to 5.

Simultaneous development of temperature and velocity profiles in the entrance region of parallel plates was first studied by Sparrow[8] for equal wall temperatures. He used Schiller's velocity profile and employed the Karman–Pohlhausen method to obtain the Nusselt number. The results reported cover a range of Prandtl numbers from 0.01 to 50.

An all-numerical finite-difference method was used by Hwang and Fan[9] for the parallel-plate combined-entry problem. Their results cover Prandtl numbers from 0.01 to 50. The same method was used by Hornbeck[10] for flow in a circular duct. He obtained the solutions for Pr = 0.7, 2, and 5.

In the present analysis, the simultaneous development of velocity and temperature profile between parallel plates in laminar flow is studied. A similarity method employing Schiller's velocity profile was used to solve the energy equation. The fluid is considered incompressible, with constant physical properties, and the wall temperatures are assumed to be uniform and equal. Longitudinal conduction and viscous dissipation are also neglected.

A closed-form expression was obtained for the Nusselt number as a function of x^* and Pr. The Nusselt numbers were compared with those obtained by more accurate methods for Pr = 0.72, 10, and 50. The results will be useful for fluids other than liquid metals.

Analysis

Consider a simultaneously developing laminar flow in the entrance region between parallel plates. The assumption of incompressible flow and constant fluid properties allows the continuity and energy equation to be written as

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{2}$$

Address reprint requests to Dr. Mortazavi at the Department of Mechanical Engineering, Isfahan University of Technology, Isfahan, Iran.

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and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

Using the Von Mises transformation ($x, y \rightarrow x, \psi$) reduces the energy equation to[9]

$$\frac{\partial T}{\partial x}=\alpha\frac{\partial}{\partial \psi}\left(u\frac{\partial T}{\partial \psi}\right) \tag{4}$$

The continuity equation is automatically satisfied by employing the stream function:

$$\psi=\int_0^y u\,dy \tag{5}$$

The familiar linear profile is approximated for u and expressed as

$$u=\frac{\tau_w(x)}{\mu}y \tag{6a}$$

Integration of Equation 5 and using Equation 6a yields

$$u=\left(\frac{2\tau_w(x)\psi}{\mu}\right)^{1/2} \tag{6b}$$

Using Equation 6b, we can write Equation 4 as

$$\frac{\partial T}{\partial s}=\frac{\partial}{\partial \psi}\left(\psi^{1/2}\frac{\partial T}{\partial \psi}\right) \tag{7a}$$

in which s is defined as

$$ds=\alpha\left(\frac{2\tau_w(x)}{\mu}\right)^{1/2}dx \tag{7b}$$

By employing a similarity variable defined as

$$\eta=\psi^{1/2}\left(\frac{4}{9s}\right)^{1/3} \tag{8}$$

we obtain the following ordinary differential equation from Equation 7a:

$$\frac{d^2T}{d\eta^2}+3\eta^2\frac{dT}{d\eta}=0 \tag{9}$$

In Equation 9, both of the convective terms of Equation 4 are included. Equation 9 can be integrated subject to the following boundary conditions:

$$\eta=0;\quad T=T_o;\qquad \eta\rightarrow\infty;\quad T=T_i \tag{10}$$

The solution is

$$\frac{T-T_o}{T_i-T_o}=\frac{1}{0.893}\int_0^\eta \exp(-\eta^3)\,d\eta \tag{11}$$

The local Nusselt number, Nu_2, may be obtained using the temperature distribution given in Equation 11

$$\mathrm{Nu}_2=\frac{-(\partial T/\partial y)y=0}{T_o-T_i}d_e$$
$$\mathrm{Nu}_2=\frac{d_e\tau_w^{1/2}(x)}{0.893\left[9\alpha\mu\int_0^x \tau_w(x)^{1/2}\,dx\right]^{1/3}} \tag{12}$$

On the other hand, the heat transfer coefficient may be defined in terms of (T_o-T_m) instead of (T_o-T_i). If we denote $(T_o-T_m$ by h_1, the corresponding Nusselt number will be

$$\mathrm{Nu}_1=\frac{h_1d_e}{K}=\frac{-\left(K\dfrac{\partial T}{\partial y}\right)_o d_e}{K(T_o-T_m)} \tag{13}$$

The dimensionless mean temperature of fluid, θ_m, is readily obtained by performing an energy balance on the fluid, which results in

$$\theta_m=1-4\int_0^{x^*}\mathrm{Nu}_2\,dx^* \tag{14}$$

The two local Nusselt numbers are related through

$$\mathrm{Nu}_1=\frac{\mathrm{Nu}_2}{\theta_m} \tag{15}$$

The average Nusselt number over length x from the entrance is given by

$$\mathrm{Nu}_{1m}=\frac{1}{4x^*}\ln\left(\frac{1}{\theta_m}\right) \tag{16}$$

Notation

a	One half channel width, m
d_e	Hydraulic diameter ($=4a$), m
K	Thermal conductivity, W/m K
Nu_1	Nusselt number defined by Equation 13, dimensionless
Nu_2	Nusselt number defined by Equation 12, dimensionless
Pr	Prandtl number, dimensionless
Re	Reynolds number, dimensionless
S	Variable defined by Equation 7, $m^3/s^{3/2}$
t	Defined by Equation 27, dimensionless
T	Temperature, K
$\bar{U}$	Uniform inlet velocity, m/s
U_1	Centerline velocity, m/s
U_1^*	Centerline velocity ($=U_1/\bar{U}$), dimensionless
u	Axial velocity component, m/s
v	Normal velocity component, m/s
x	Axial distance, m
x^+	Defined as $x/(d_e\mathrm{Re})$, dimensionless
x^*	Defined as $x/(d_e\mathrm{Re\,Pr})$, dimensionless
y	Normal distance, m

Greek letters

α	Thermal diffusivity, m^2/s
β	Defined by Equation 1, m^{-1}
δ	Velocity boundary layer thickness, m
δ_t	Thermal boundary layer thickness, m
η	Similarity parameter (Equation 8), dimensionless
μ	Viscosity, $N\cdot s/m^2$
ν	Kinematic viscosity, m^2/s
ρ	Density, kg/m^3
τ_w	Wall shear stress, N/m^2
θ	Dimensionless temperature $[=(T-T_o)/(T_i-T_o)]$

Subscripts

o	Wall condition
i	Inlet condition
m	Mean value

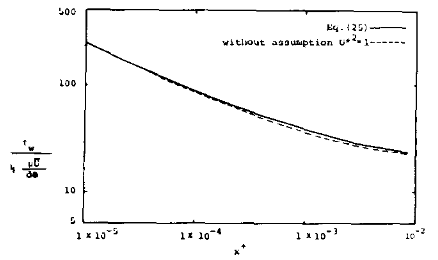

Figure 1 Distribution of wall shear stress along the flow channel

The main task remaining is to develop an appropriate expression for the wall shear stress, which will lead to a closed-form expression for the Nusselt number when substituted in Equation 12. A simple and convenient expression for the velocity profile in the boundary layer was first used by Schiller.[3] It gives reasonably good agreement with Schlichting's more exact result:

$$u = U_1\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] 0 < y < \delta \tag{17}$$

This equation satisfies the conditions

$$\begin{aligned} u &= 0; \quad y = 0 \\ u &= U_1; \quad y = \delta \\ \frac{\partial u}{\partial y} &= 0 \end{aligned} \tag{18}$$

The velocity distribution in Equation 18 allows us to obtain the following results from the continuity and momentum integral equations[8]:

$$dx = \frac{3}{10}\frac{a^2}{\nu}\frac{(U_1 - \bar{U})(9U_1 - 7\bar{U})}{U_1^2}\,dU_1 \tag{19}$$

and

$$\frac{\delta}{a} = 3\left(1 - \frac{\bar{U}}{U_1}\right) \tag{20}$$

In nondimensional form, Equation 19 is written as

$$dx^+ = \frac{3}{160}\frac{(U_1^* - 1)(9U_1^*/7)}{U_1^{*2}}\,dU_1^* \tag{21}$$

It is clear that such an expression does not satisfy the condition of fully developed flow because dU_1^*/dx^+ does not approach zero as $U_1^* \to 1.5$. Equation 21 may be integrated to yield

$$x^+ = \frac{3}{160}\left(9U_1^* - 16\ln U_1^* - \frac{7}{U_1^*} - 2\right) \tag{22}$$

It is possible to obtain a closed-form expression for Nu_2 by using Equation 22. However, to derive an explicit form of solution for Nu_1, we will now use the approximation of $U_1^{*2} \simeq 1$, which is only valid close to the entrance. Equation 22 then becomes

$$U_1^* - 1 = \sqrt{\frac{160x^+}{3}} \tag{23}$$

The shear stress at the wall is defined as

$$\tau_w = \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} = 2\mu\frac{U_1}{\delta} \tag{24}$$

Employing Equations 20 and 23 yields

$$\tau_w = \frac{2\mu\bar{U}}{3a}\frac{\left(1 + \sqrt{\frac{160}{3}x^+}\right)^2}{\sqrt{\frac{160}{3}x^+}} \tag{25}$$

Substituting for τ_w in Equation 12 and integrating gives

$$Nu_2 = 0.242\frac{\left(1 + \frac{\sqrt{160}}{\sqrt{3}}\right)Pr^{1/2}x^{*^{1/2}}}{\left(\frac{1}{3} + \frac{1}{5}\frac{\sqrt{160}}{\sqrt{3}}Pr^{1/2}x^{*^{1/2}}\right)^{1/3}}x^{*^{1/2}}Pr^{-1/6} \tag{26}$$

in which x^* is the reciprocal of the Graetz number.

In order to obtain an expression for Nu_1, we must specify the dimensionless mean temperature. Substituting Equation 26 into Equation 14 and integrating gives

$$\theta_m = 1 - 1.3276944\,Pr^{-2/3}(3t^5 - t^2) \tag{27}$$

where

$$t = \left[\frac{1}{3} + \frac{1}{5}\left(\frac{160}{3}x^*Pr\right)^{1/2}\right]^{1/3} \tag{28}$$

For the purpose of comparing this expression with the more exact results available in the literature, Nu_{1m} has to be determined. Combining Equations 16 and 27 yields

$$Nu_{1m} = \frac{1}{4x^*}\ln[1 - 1.3276944\,Pr^{-2/3}(3t^5 - t^2)]^{-1} \tag{29}$$

The seven significant digits in Equations 27 and 29 reflect the fact that (a) close to the entrance, θ_m is very close to 1.0, and (b) as indicated by Equation 16, a small error in θ_m will be pronounced when Nu_{1m} is calculated.

Results and discussion

The results of calculation are shown in Figures 1–4. Figure 1 compares the approximate distribution of the nondimensional wall shear stress given by Equation 25 to the more accurate results obtained by the combination of Equations 20, 22, and 24.

The results used for the comparison of the Nusselt numbers are those of Hwang and Fan,[9] which are given in the monograph by Shah and London.[1] These results were obtained from an all-numerical finite-difference solution of the combined entry problem for parallel plates with equal temperatures. The values of Nu_{1m} were tabulated and also graphically represented on page 191 of Ref. 1 for Pr=0.1, 0.72, 10, and 50. As indicated by Equation 29, the Nusselt number in a simultaneously developing flow depends on both x^* and Pr. Hence the representation of Equation 29 is for three different values of Pr: 0.72, 10, 50, and excluding Pr=0.1. As explained before,

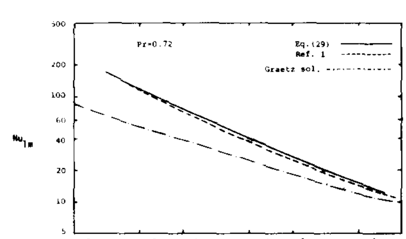

Figure 2 Average Nusselt number (Nu_{1m}) for Pr=0.72

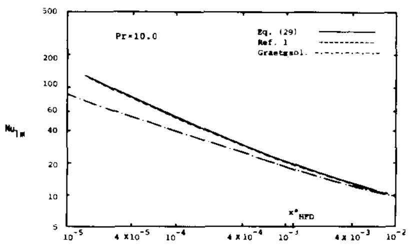

Figure 3 Average Nusselt number (Nu_{1m}) for Pr = 10.0

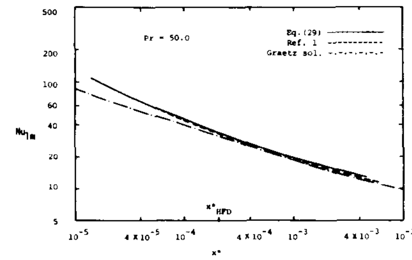

Figure 4 Average Nusselt number (Nu_{1m}) for Pr = 50.0

the present analysis is not applicable to fluids with low Prandtl numbers because the approximation of linear velocity profile in the thermal boundary layer breaks down. The results of Equation 29 and the more exact results of Hwang and Fan are presented in Figures 2–4 for Pr = 0.72, 10, and 50.

Also indicated in these figures are the values of the Nusselt number for the hydrodynamically developed and thermally developing flow. These values were calculated by Shah,[11] using 121 terms of the Graetz series for $x^* > 10^{-4}$. The values for $x^* < 10^{-4}$ were determined from the extended Lévêque solution. In a thermally developing flow the Nusselt number is independent of the Prandtl number, so its representations in Figures 2–4 are identical. These figures show that this result approaches the more exact result indicated by the dashed line as Pr increases. In fact, for Pr = 50, the difference is very small, even at low values of x^*. It is small because of rapid development of the velocity boundary layer, compared to that of the thermal boundary layer, for fluids with high values of Pr. This condition justifies the assumption of hydrodynamically fully developed flow.

Figures 2–4 also indicate reasonable agreement between the predictions of Equation 29 and the more exact results reported in Ref. 1. This good agreement for the heat transfer prediction exists even after the velocity profile becomes parabolic, despite the inaccuracy inherent in the chosen velocity profile, Equation 17, in this region. The hydrodynamic entry length represented in Figures 2–4 by x^*_{HFD} were calculated from the criterion of $x^+_{HFD} = 0.011$. This value for the beginning of the hydrodynamically fully developed flow was used by Bodoia.[12]

Generally speaking, Equation 29 predicts values that are higher than the more exact results of Ref. 1. The difference depends on the values of Pr and x^*. The error increases as Pr decreases, which is related to the unjustifiable approximation of linear velocity profile used in the energy equation. In addition, the larger disagreements resulting from increasing x^* are attributed to the inaccurate representation of the parabolic velocity profile by the second-degree polynomial used in this analysis to determine the shear stress at the wall.

Finally, due to two main approximations used in this analysis, the Nusselt number predicted by Equation 29 does not approach the fully developed value of 7.54 as $x^* \to \infty$. Hence Equation 29 is not recommended for use with an x^* larger than the values shown in Figures 2–4. Moreover, the present analysis is not suitable for fluids with low Prandtl number, such as liquid metals.

Conclusions

A closed-form expression was obtained for the Nusselt number in simultaneously developing flow between parallel plates with the same uniform wall temperature but different from the fluid inlet temperature. The energy equation was solved by similarity methods, and a second-degree polynomial was employed for the velocity profile in order to determine the shear stress at the wall.

The prediction by the present method was compared to existing, more exact results obtained by an all-numerical method. Reasonable agreement was observed. It is recommended that the prediction method described here be used to predict the heat transfer in a simultaneously developing laminar flow of gases and liquids—other than liquid metals—between parallel plates at the same uniform temperature.

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